Perimeter Variance of Uniform Random Triangles

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Abstract. Let T be a random triangle in a disk D of radius R (meaning that vertices are independent and uniform in D). We determine the bivariate density for two arbitrary sides a, b of T. In particular, we compute that $\mathrm{E}(a\,b) = (0.837...)R^2$, which implies that $\mathrm{Var}(\text{perimeter}) = (0.649...)R^2$. No closed-form expression for either coefficient is known. The Catalan numbers also arise here.

Let A, B, C denote three independent uniformly distributed points in the disk $D = \{(\xi, \eta) : \xi^2 + \eta^2 \leq R^2\}$. Let T denote the triangle with sides a, b, c opposite the vertices A, B, C. We are interested in the perimeter $a + b + c$ of triangle T. The univariate density $f(x)$ for side a is [1, 2, 3, 4, 5, 6, 7]

$$\frac{4x}{\pi R^2} \arccos\left(\frac{x}{2R}\right) - \frac{x^2}{\pi R^4}\sqrt{4R^2 - x^2}, \qquad 0 < x < 2R$$

and

$$\mathrm{E}(a) = \frac{128}{45\pi} R = (0.9054147873672267990407609...)R, \qquad \mathrm{E}(a^2) = R^2.$$

Clearly

$$\mathrm{E}(\text{perimeter}) = 3\,\mathrm{E}(a) = \frac{128}{15\pi} R = (2.7162443621016803971222828...)R$$

but to compute $\mathrm{Var}(\text{perimeter}) = \mathrm{E}(\text{perimeter}^2) - \mathrm{E}(\text{perimeter})^2$, we will further need to consider cross-correlation ρ between sides.

The bivariate density $f(x, y)$ for sides a, b is

$$f(x, y) = \begin{cases} \varphi(x, y) & \text{if } x + y \leq 2R, \\ \psi(x, y) & \text{if } x + y > 2R \text{ and } x \leq 2R \end{cases}$$

when $0 \leq y \leq x$ (use symmetry otherwise) where

$$\begin{aligned} \varphi(x, y) \;=\; & \frac{2xy}{\pi R^6}\left\{-\sqrt{(2R - x - y)(x - y)(2R + x - y)(x + y)} + \right. \\ & \left. 2(R - y)^2 \arccos\left(\frac{x^2 - 2Ry + y^2}{2x(R - y)}\right) + 2R^2 \arccos\left(\frac{x^2 + 2Ry - y^2}{2Rx}\right)\right\} + \\ & \frac{8xy}{\pi^2 R^6} \int\limits_{R-y}^{R} t \arccos\left(\frac{t^2 + x^2 - R^2}{2tx}\right) \arccos\left(\frac{t^2 + y^2 - R^2}{2ty}\right) dt, \end{aligned}$$

$$\psi(x,y)=\frac{8xy}{\pi^2R^6}\int\limits_{x-R}^{R}t\arccos\left(\frac{t^2+x^2-R^2}{2tx}\right)\arccos\left(\frac{t^2+y^2-R^2}{2ty}\right)dt.$$

It follows by numerical integration that

$$\mathrm{E}(a\,b)=(0.8378520652962219016710654...)R^2$$

hence

$$\rho(a,b)=\frac{\mathrm{E}(a\,b)-\mathrm{E}(a)\,\mathrm{E}(b)}{\sqrt{\mathrm{Var}(a)\,\mathrm{Var}(b)}}=0.1002980835659001715822627...,$$

$$\mathrm{E}(\text{perimeter}^2)=3\,\mathrm{E}(a^2)+6\,\mathrm{E}(a\,b)=(8.0271123917773314100263929...)R^2,$$

$$\mathrm{Var}(\text{perimeter})=(0.6491289571281667551974101...)R^2.$$

Exact evaluation of $\mathrm{E}(a\,b)$ remains an open problem. We review derivation of the univariate case in section 1, imitating the analysis in [8, 9] very closely. (Parry's thesis [8] is concerned with triangles in three-dimensional space; it is surprising that our two-dimensional analog has not yet been examined.) The bivariate case is covered in section 2. An experimental consequence of our work is the formula

$$\mathrm{E}(a^2\,b^2)=\frac{13}{12}R^4$$

which we prove via a different approach in section 3. Finally, in section 4, the Catalan numbers from combinatorics appear rather unexpectedly.

1. Univariate Case

We omit geometric details, referring to [8, 9] instead. The distance t between point C and the origin has density $2t/R^2$ for $0<t<R$. Let $f(x\,|\,t)$ be the conditional density for distance x between points C and B, given t. We will compute the sought-after density $f(x)$ for side a via

$$f(x)=\int\limits_0^R f(x\,|\,t)\frac{2t}{R^2}dt.$$

There are two subcases.

1.1. $0<x<R$**.**

$$f(x)=\int\limits_0^{R-x}\frac{2x}{R^2}\frac{2t}{R^2}dt+\int\limits_{R-x}^{R}\frac{2x}{\pi R^2}\arccos\left(\frac{t^2+x^2-R^2}{2tx}\right)\frac{2t}{R^2}dt$$

which corresponds to formula (1.12) in Parry's thesis [8]. The arccos term arises since, if the portion of a circle of radius x, center C contained within D has arclength $2\theta x$, then $f(x\,|\,t)=(2\theta x)/(\pi R^2)$; the Law of Cosines gives θ.

1.2. $R < x < 2R$**.**

$$f(x) = \int_{x-R}^{R} \frac{2x}{\pi R^2} \arccos\left(\frac{t^2+x^2-R^2}{2tx}\right)\frac{2t}{R^2}dt$$

which corresponds to Parry's (1.18). Straightforward integration provides the desired result (valid in both of the preceding regions).

2. Bivariate Case

We omit geometric details, referring to [8] instead. The distance t between point C and the origin has density $2t/R^2$ for $0 < t < R$. Let $f(x, y \,|\, t)$ be the conditional density for distance x between points B and C, and distance y between points A and C, given t. We will compute the sought-after density $f(x, y)$ for sides a, b via

$$f(x,y) = \int_0^R f(x,y\,|\,t)\frac{2t}{R^2}dt.$$

There are six subcases.

2.1. $y < x$ **and** $0 < x < R$**.**

$$\begin{aligned} f(x,y) \quad &= \quad \int_0^{R-x} \frac{2x}{R^2}\frac{2y}{R^2}\frac{2t}{R^2}dt + \int_{R-x}^{R-y} \frac{2x}{\pi R^2}\arccos\left(\frac{t^2+x^2-R^2}{2tx}\right)\frac{2y}{R^2}\frac{2t}{R^2}dt + \\ &\quad \int_{R-y}^{R} \frac{2x}{\pi R^2}\arccos\left(\frac{t^2+x^2-R^2}{2tx}\right)\frac{2y}{\pi R^2}\arccos\left(\frac{t^2+y^2-R^2}{2ty}\right)\frac{2t}{R^2}dt \end{aligned}$$

which corresponds to formula (4.26) in Parry's thesis [8].

2.2. $R < x < 2R$ **and** $0 < y < 2R - x$**.**

$$\begin{aligned} f(x,y) \quad &= \quad \int_{x-R}^{R-y} \frac{2x}{\pi R^2}\arccos\left(\frac{t^2+x^2-R^2}{2tx}\right)\frac{2y}{R^2}\frac{2t}{R^2}dt + \\ &\quad \int_{R-y}^{R} \frac{2x}{\pi R^2}\arccos\left(\frac{t^2+x^2-R^2}{2tx}\right)\frac{2y}{\pi R^2}\arccos\left(\frac{t^2+y^2-R^2}{2ty}\right)\frac{2t}{R^2}dt \end{aligned}$$

which corresponds to Parry's (4.29). Straightforward integration gives $\varphi(x, y)$ (valid in both of the preceding regions).

2.3. $R < x < 2R$ **and** $2R - x < y < x$**.**

$$f(x,y) = \int\limits_{x-R}^{R} \frac{2x}{\pi R^2} \arccos\left(\frac{t^2+x^2-R^2}{2tx}\right) \frac{2y}{\pi R^2} \arccos\left(\frac{t^2+y^2-R^2}{2ty}\right) \frac{2t}{R^2} dt$$

which corresponds to Parry's (4.32). This is, of course, $\psi(x,y)$.

2.4. $x < y$ **and** $0 < y < R$**.**

$$\begin{aligned} f(x,y) &= \int\limits_{0}^{R-y} \frac{2x}{R^2}\frac{2y}{R^2}\frac{2t}{R^2} dt + \int\limits_{R-y}^{R-x} \frac{2x}{R^2}\frac{2y}{\pi R^2} \arccos\left(\frac{t^2+y^2-R^2}{2ty}\right) \frac{2t}{R^2} dt + \\ & \int\limits_{R-x}^{R} \frac{2x}{\pi R^2} \arccos\left(\frac{t^2+x^2-R^2}{2tx}\right) \frac{2y}{\pi R^2} \arccos\left(\frac{t^2+y^2-R^2}{2ty}\right) \frac{2t}{R^2} dt \end{aligned}$$

which corresponds to Parry's (4.35). This is, of course, $\varphi(y,x)$.

2.5. $R < y < 2R$ **and** $0 < x < 2R - y$**.**

$$\begin{aligned} f(x,y) &= \int\limits_{y-R}^{R-x} \frac{2x}{R^2}\frac{2y}{\pi R^2} \arccos\left(\frac{t^2+y^2-R^2}{2ty}\right) \frac{2t}{R^2} dt + \\ & \int\limits_{R-x}^{R} \frac{2x}{\pi R^2} \arccos\left(\frac{t^2+x^2-R^2}{2tx}\right) \frac{2y}{\pi R^2} \arccos\left(\frac{t^2+y^2-R^2}{2ty}\right) \frac{2t}{R^2} dt \end{aligned}$$

which corresponds to Parry's (4.38). This is, of course, $\varphi(y,x)$.

2.6. $R < y < 2R$ **and** $2R - y < x < y$**.**

$$f(x,y) = \int\limits_{y-R}^{R} \frac{2x}{\pi R^2} \arccos\left(\frac{t^2+x^2-R^2}{2tx}\right) \frac{2y}{\pi R^2} \arccos\left(\frac{t^2+y^2-R^2}{2ty}\right) \frac{2t}{R^2} dt$$

which corresponds to Parry's (4.41). This is, of course, $\psi(y,x)$.

3. Characteristic Function

We follow an approach found in [10, 11]. Let u, v, w denote the squared distances between A, B, C and the origin O. Let φ denote the angle between vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and ψ denote the angle between vectors $\overrightarrow{OA}, \overrightarrow{OC}$. We have

$$a^2 = v + w - 2\sqrt{vw}\cos(\psi - \varphi),$$

$$b^2 = u + w - 2\sqrt{uw}\cos(\psi),$$
$$c^2 = u + v - 2\sqrt{uv}\cos(\varphi)$$

by the Law of Cosines, where u, v, w are independent uniform on $[0, R^2]$ and φ, ψ are independent uniform on $[0, 2\pi]$. The characteristic function for (a^2, b^2, c^2) is thus

$$\begin{aligned} g(r,s,t) &= \frac{1}{R^6}\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} d\varphi\, d\psi \int_0^{R^2}\int_0^{R^2}\int_0^{R^2} du\, dv\, dw\ \exp\Big[ir\left(v + w - 2\sqrt{vw}\cos(\psi - \varphi)\right) + \\ &\qquad is\left(u + w - 2\sqrt{uw}\cos(\psi)\right) + it\left(u + v - 2\sqrt{uv}\cos(\varphi)\right)\Big]. \end{aligned}$$

It is well-known that

$$\mathrm{E}(c^2) = \frac{1}{i}\left.\frac{\partial g}{\partial t}\right|_{r=s=t=0}, \qquad \mathrm{E}(b^2\, c^2) = \frac{1}{i^2}\left.\frac{\partial^2 g}{\partial s\, \partial t}\right|_{r=s=t=0}$$

and the former becomes

$$\begin{aligned} \mathrm{E}(c^2) &= \left.\frac{1}{i}\frac{\partial}{\partial t}\frac{1}{R^4}\frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{R^2}\int_0^{R^2} du\, dv \exp\Big[it\left(u + v - 2\sqrt{uv}\cos(\varphi)\right)\Big]\right|_{t=0} \\ &= \left.\frac{1}{i}\frac{\partial}{\partial t}\frac{1}{R^4}\int_0^{R^2}\int_0^{R^2} \exp\left(it\left(u+v\right)\right) J_0\left(2t\sqrt{uv}\right) du\, dv\right|_{t=0} \end{aligned}$$

where $J_0(\theta)$ is the zeroth Bessel function of the first kind; hence

$$\begin{aligned} \mathrm{E}(c^2) &= \frac{1}{i}\frac{1}{R^4}\int_0^{R^2}\int_0^{R^2} \left.\frac{\partial}{\partial t}\exp\left(it\left(u+v\right)\right) J_0\left(2t\sqrt{uv}\right)\right|_{t=0} du\, dv \\ &= \frac{1}{i}\frac{1}{R^4}\int_0^{R^2}\int_0^{R^2} i(u+v) du\, dv = R^2 \end{aligned}$$

which is consistent with before. The latter becomes

$$\begin{aligned} \mathrm{E}(b^2\, c^2) &= \frac{1}{i^2}\frac{\partial^2}{\partial s\, \partial t}\frac{1}{R^6}\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} d\varphi\, d\psi \int_0^{R^2}\int_0^{R^2}\int_0^{R^2} du\, dv\, dw \exp\Big[is\left(u + w - 2\sqrt{uw}\cos(\psi)\right) + \\ &\qquad \left. it\left(u + v - 2\sqrt{uv}\cos(\varphi)\right)\Big]\right|_{s=t=0} \\ &= \left. -\frac{\partial^2}{\partial s\, \partial t}\frac{1}{R^6}\int_0^{R^2}\int_0^{R^2}\int_0^{R^2} \exp\left(is\left(u+w\right)\right) J_0\left(2s\sqrt{uw}\right) \exp\left(it\left(u+v\right)\right) J_0\left(2t\sqrt{uv}\right) du\, dv\, dw\right|_{s=t=0} \end{aligned}$$

$$= -\frac{1}{R^6}\int_0^{R^2}\int_0^{R^2}\int_0^{R^2}\frac{\partial^2}{\partial s\,\partial t}\exp\left(is\left(u+w\right)\right)J_0\left(2s\sqrt{uw}\right)\exp\left(it\left(u+v\right)\right)J_0\left(2t\sqrt{uv}\right)\Bigg|_{s=t=0}du\,dv\,dw$$

$$= -\frac{1}{R^6}\int_0^{R^2}\int_0^{R^2}\int_0^{R^2}-(u+v)(u+w)du\,dv\,dw=\frac{13}{12}R^4$$

as was to be shown. The fact that $13/12-1=1/12\neq 0$ offers the simplest proof we know that arbitrary sides of a random triangle in D must be dependent.

4. Catalan Numbers

Let $R=1$ for the remainder of our discussion. From

$$\mathrm{P}\left(a^2<x\right)=\mathrm{P}\left(a<\sqrt{x}\right)=\int_0^{\sqrt{x}}\left(\frac{4\xi}{\pi}\arccos\left(\frac{\xi}{2}\right)-\frac{\xi^2}{\pi}\sqrt{4-\xi^2}\right)d\xi$$

we obtain that the density for a^2 is

$$\left(\frac{4\sqrt{x}}{\pi}\arccos\left(\frac{\sqrt{x}}{2}\right)-\frac{x}{\pi}\sqrt{4-x}\right)\frac{1}{2\sqrt{x}},\qquad 0<x<4.$$

On the one hand, the characteristic function for a^2 is

$$\int_0^1\int_0^1\exp\left(it\left(u+v\right)\right)J_0\left(2t\sqrt{uv}\right)du\,dv$$

by the preceding section; on the other hand, it is

$$\begin{aligned}&\int_0^4\exp(itx)\left(\frac{4\sqrt{x}}{\pi}\arccos\left(\frac{\sqrt{x}}{2}\right)-\frac{x}{\pi}\sqrt{4-x}\right)\frac{1}{2\sqrt{x}}dx\\ =\ &\frac{i}{t}\left[1-\exp(2it)\left(J_0(2t)-iJ_1(2t)\right)\right]\\ =\ &\frac{i}{t}\left[1-h(t)\right]\end{aligned}$$

where $J_1(\theta)=-J_0'(\theta)$. A direct evaluation of the double integral seems to be difficult. Boersma [12], using work of Zernike & Nijboer [13, 14, 15], gave a rapidly-convergent series for the inner integral:

$$\int_0^1\exp\left(itu\right)J_0\left(2t\sqrt{uv}\right)du=\frac{\sqrt{\pi}}{t^{3/2}v^{1/2}}\exp\left(\frac{it}{2}\right)\sum_{n=0}^{\infty}(-i)^n(2n+1)J_{n+1/2}\left(\frac{t}{2}\right)J_{2n+1}\left(2t\sqrt{v}\right)$$

but this apparently does not help with the outer integral.

Let $I_0(\theta)$ be the zeroth modified Bessel function of the first kind and $I_1(\theta) = I_0'(\theta)$. We note that the exponential generating function for the Catalan numbers [16]:

$$\exp(2t)\left(I_0(2t) - I_1(2t)\right) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\binom{2n}{n} t^n$$

is remarkably similar to the expression for $h(t)$. Replacing t by it, we obtain

$$h(t) = \exp(2it)\left(J_0(2t) - iJ_1(2t)\right) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\binom{2n}{n} (it)^n$$

because $J_0(i\theta) = I_0(\theta)$, $J_1(i\theta) = iI_1(\theta)$. Therefore the Catalan numbers are associated with the characteristic function for a^2. We wonder if a two-dimensional integer array, suitably generalizing the Catalan numbers, can be associated with the characteristic function for (a^2, b^2):

$$\int_0^1 \int_0^1 \int_0^1 \exp\left(is\left(u+w\right)\right) J_0\left(2s\sqrt{uw}\right) \exp\left(it\left(u+v\right)\right) J_0\left(2t\sqrt{uv}\right) du\, dv\, dw.$$

Since the bivariate density $f(x, y)$ for (a, b) is much more complicated than the univariate density $f(x)$ for a, an answer to our question may be a long time coming.

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